

ON A CONJECTURE OF H. GUPTA

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ABSTRACT. Denote by $r(n)$ the length of a shortest integer sequence on a circle containing all permutations of the set $\{1, 2, \dots, n\}$ as subsequences. Hansraj Gupta conjectured in 1981 that $r(n) \leq \frac{n^2}{2}$. In this paper, we confirm the conjecture for the case where n is even and show that $r(n) < \frac{n^2}{2} + \frac{n}{4} - 1$ if n is odd.

0. INTRODUCTION

Let $n > 1$ be a positive integer. One would like to find the smallest number $r = r(n)$ for which there exists a sequence x_1, \dots, x_r such that every permutation of the first n natural numbers is a subsequence of $x_j, x_{j+1}, \dots, x_r, x_1, x_2, \dots, x_{j-1}$ for some j .

In other words, if we write the sequence x_1, \dots, x_r on the circumference of a circle, then every permutation of the set $\{1, 2, \dots, n\}$ can be found starting from a point on the circle, moving in the clockwise direction and never turning back nor crossing the starting point. Such a sequence is called a *rosary* of degree n . Examples of rosaries of degrees 2, 3, 4 and 5 are given in Figure 1.

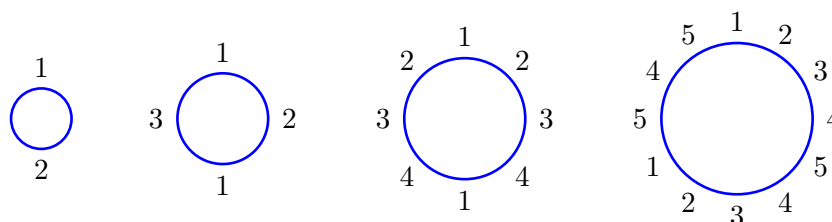


FIGURE 1. Examples of rosaries.

It is easy to show that $r(n) \leq n^2 - 3n + 4$, since the following sequence

$$1(2, \dots, n)_{n-2} 2 = 1, \underbrace{2, \dots, n, 2, \dots, n, \dots, 2, \dots, n, 2}_{(n-2) \text{ times}}$$

is a rosary of degree n . Indeed, for any permutation a_1, \dots, a_n starting with 1, either a_2, \dots, a_n is a concatenation of at most $n - 2$ increasing strings, or the permutation is $1, n, n - 1, \dots, 2$.

The following conjecture was suggested by Hansraj Gupta in [3]. It can also be found in the well-known book [4] under Problem E22.

Conjecture (H. Gupta). *The inequality $r(n) \leq \frac{n^2}{2}$ is satisfied for all integers $n > 1$.*

Here is the structure of the present paper:

- 1) The asymptotic formula $r(n) \sim \frac{n^2}{2}$ as $n \rightarrow +\infty$ is established by applying a result from [6] and the results of Sections 2 and 4.
- 2) For any even n , a rosary of length $\frac{n^2}{2}$ is constructed.
- 3) Examples where $r(n) < \frac{n^2}{2}$ are given.
- 4) For any odd n , we construct a rosary of length less than $\frac{n^2}{2} + \frac{n}{4} - 1$.
- 5) We give examples showing that the desired bound $\frac{n^2}{2}$ is not reachable by our method if n is odd.

1. THE ASYMPTOTIC BEHAVIOUR OF $r(n)$

According to H. Gupta, his conjecture was motivated by the problem of constructing a shortest sequence (on a straight line) which contains all permutations of the numbers $1, 2, \dots, n$ as subsequences. The latter was first suggested by R. M. Karp (as mentioned in the paper [2]).

Denote by $s(n)$ the length of such a sequence. The inequality $s(n) \leq n^2 - 2n + 4$ was provided by several people through explicit constructions (see, for instance, [1, 5, 7]).

In 1975, D. J. Kleitman and D. J. Kwiatkowski [6] established the lower bounds

$$n^2 - C_\varepsilon n^{7/4+\varepsilon} < s(n),$$

for any real number $\varepsilon > 0$ and a constant C_ε depending on ε .

Theorem 1. *The function r is asymptotic to $\frac{n^2}{2}$ as integer n tends to $+\infty$.*

Proof. By Theorems 2 and 3, the inequality $r(n) < \frac{n^2}{2} + \frac{n}{4}$ is satisfied for any natural $n > 1$. On the other hand, the function r is bounded below:

$$s(n) \leq 2r(n).$$

Indeed, if a_1, \dots, a_n is a rosary of degree n , then the sequence $a_1, a_2, \dots, a_n, a_1, a_2, \dots, a_n$ contains all permutations of the first n natural numbers as subsequences.

Finally, using the result of Kleitman and Kwiatkowski, we obtain

$$\frac{n^2}{2} - C n^{15/8} < r(n) < \frac{n^2}{2} + \frac{n}{4}$$

for a constant $C > 0$. Therefore, $r(n) \sim \frac{n^2}{2}$ as $n \rightarrow +\infty$. □

2. A PROOF FOR n EVEN

By default all sequences we deal with here are cyclic (one may imagine that they are written on a circle). A cyclic sequence will be called a *cycle* for brevity. If we want to emphasize that a_1, \dots, a_n is considered as an ordinary sequence (and not a cycle), then it will be called a *string*.

A *block* of a sequence (cycle or string) is a subsequence consisting of consecutive elements. For instance, (4, 5, 3) and (2, 1, 7, 4) are blocks of the cycle 7, 4, 5, 3, 6, 2, 1. A *maximal increasing block* is an increasing block $a_{j+1} < a_{j+2} < \dots < a_{j+k}$ such that $a_j > a_{j+1}$ and $a_{j+k} > a_{j+k+1}$. We analogously define a *maximal decreasing block*. In the example above the blocks (7, 4), (5, 3) and (6, 2, 1) are maximal decreasing, and the blocks (4, 5), (3, 6), (2) and (1, 7) are maximal increasing.

The *code* of a cycle a_1, \dots, a_n is the following cyclic sequence

$$H(a_2 - a_1), H(a_3 - a_2), \dots, H(a_n - a_{n-1}), H(a_1 - a_n),$$

where H denotes the Heaviside step function, $H(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$ Thus, the code of the cycle 7, 4, 5, 3, 6, 2, 1 is 0, 1, 0, 1, 0, 0, 1.

The *code* of a string a_1, \dots, a_n is the following string

$$H(a_2 - a_1), H(a_3 - a_2), \dots, H(a_n - a_{n-1}).$$

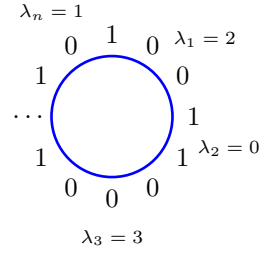
Note that the code of a string of length n is a string of length $(n - 1)$.

Consider a permutation a_1, \dots, a_n of the first n natural numbers regarded as a cyclic sequence. Let the cycle c_1, \dots, c_n be its code. Suppose that the code has x ones $c_{i_1} = c_{i_2} = \dots = c_{i_x} = 1$ and y zeros, where $x + y = n$.

Denote by λ_1 the number of zeros between c_{i_1} and c_{i_2} , by λ_2 the number of zeros between c_{i_2} and c_{i_3} , etc., by λ_x the number of zeros between c_{i_x} and c_{i_1} . We have

$$\lambda_1 + \lambda_2 + \dots + \lambda_x = y.$$

It is clear that the integers $(\lambda_i + 1)$ are lengths of the maximal decreasing blocks of the cycle a_1, \dots, a_n .



Lemma 1. *Let K and M be positive integers. If there exists an index $i \in [1, x]$ such that*

$$\lambda_{i+1} + \lambda_{i+2} + \dots + \lambda_{i+K} \geq y - M + 1,$$

then the cycle a_1, \dots, a_n is a subsequence of the cycle¹

$$(1, 2, \dots, n)_{M-1} (1, 2, \dots, n-1) (n, \dots, 2, 1)_{K-1} (n, \dots, 3, 2).$$

Proof. Without loss of generality (up to a cyclic permutation), we may suppose that the code of the cycle a_1, \dots, a_n has the following form

$$c_1, \dots, c_n = 1(0)^{\lambda_1} 1(0)^{\lambda_2} \dots 1(0)^{\lambda_{x-K}} 1(0)^{\lambda_{x-K+1}} \dots 1(0)^{\lambda_x},$$

where

$$\lambda_{x-K+1} + \lambda_{x-K+2} + \dots + \lambda_x \geq y - M + 1,$$

and so

$$\lambda_1 + \lambda_2 + \dots + \lambda_{x-K} \leq M - 1.$$

A part $(0)^{\lambda_1} = \underbrace{0, 0, \dots, 0}_{\lambda}$ of the code corresponds to inequalities $a_{\bullet} > a_{\bullet+1} > \dots > a_{\bullet+\lambda} < a_{\bullet+\lambda+1}$. On one hand, this gives λ increasing blocks $(a_{\bullet+1}), \dots, (a_{\bullet+\lambda-1}), (a_{\bullet+\lambda} < a_{\bullet+\lambda+1})$, and on the other hand, one decreasing block $a_{\bullet} > a_{\bullet+1} > \dots > a_{\bullet+\lambda}$. This means that the sequence a_1, \dots, a_n is a juxtaposition of

$$1 + \lambda_1 + \lambda_2 + \dots + \lambda_{x-K} \leq M$$

increasing blocks, the first one being $(a_1 < a_2)$, and K decreasing blocks encoded by

$$(0)^{\lambda_{x-K+1}} 1(0)^{\lambda_{x-K+2}} \dots 1(0)^{\lambda_x}.$$

Hence, it is a subsequence of $(1, 2, \dots, n)_{M-1} (1, 2, \dots, n-1) (n, \dots, 2, 1)_{K-1} (n, \dots, 3, 2)$. \square

The following theorem shows that $r(n) \leq \frac{n^2}{2}$ for any even positive integer n .

Theorem 2. *Let k be a positive integer.*

a) *If $n = 4k$, then the sequence*

$$(1, 2, \dots, n)_k (1, n, n-1, \dots, 2)_k \tag{1}$$

is a rosary of degree n .

¹We write $(x_1, x_2, \dots, x_m)_k$ for the juxtaposition of k copies of the sequence in the brackets.

b) If $n = 4k + 2$, then the sequence

$$(1, 2, \dots, n)_{k+1} (1, n, n-1, \dots, 2)_k \quad (2)$$

is a rosary of degree n .

Proof. **a)** Assume that a permutation a_1, \dots, a_n of the numbers $1, 2, \dots, n$ does not occur in (1). By Lemma 1 with $K = M = k$,

$$\lambda_{i+1} + \lambda_{i+2} + \dots + \lambda_{i+k} \leq y - k, \quad \text{for any } 1 \leq i \leq x.$$

Summing all x inequalities leads to

$$\begin{aligned} ky &= k(\lambda_1 + \lambda_2 + \dots + \lambda_x) \\ &= (\lambda_1 + \lambda_2 + \dots + \lambda_k) + (\lambda_2 + \lambda_3 + \dots + \lambda_{k+1}) + \dots + (\lambda_x + \lambda_1 + \dots + \lambda_{k-1}) \\ &\leq x(y - k). \end{aligned}$$

Making the substitutions $y = n - x$ and $n = 4k$, one obtains

$$\begin{aligned} k(n - x) &\leq x(-x + n - k), \\ x^2 - nx + kn &\leq 0, \\ (x - 2k)^2 &\leq 0. \end{aligned}$$

Hence, $x = 2k$ and $y = 2k$, and all the inequalities are actually equalities. In particular,

$$\lambda_{i+1} + \lambda_{i+2} + \dots + \lambda_{i+k} = y - k = k. \quad \text{for any } 1 \leq i \leq x. \quad (3)$$

Let us show that in this case the cycle a_1, \dots, a_n is a subsequence of (1). Without loss of generality, we may take $a_1 = 1$, implying also that $c_n = 0$ and $c_1 = 1$ (as $a_n > a_1 < a_2$).

First, the code of the string a_1, \dots, a_{2k+1} equals the part $1(0)^{\lambda_1} \dots 1(0)^{\lambda_k}$ of the sequence c_1, \dots, c_n , and occurs in the string

$$(1, n, n-1, \dots, 2)_k,$$

since $1(0)^\lambda$ encodes inequalities $a_\bullet < a_{\bullet+1} > a_{\bullet+2} > \dots > a_{\bullet+\lambda} > a_{\bullet+\lambda+1}$, and thus the string a_2, \dots, a_{2k+1} consists of k maximal decreasing blocks.

Second, the remaining string

$$a_{2k+2}, a_{2k+3}, \dots, a_{4k}$$

has the code $(0)^{\lambda_{k+1}} 1(0)^{\lambda_{k+2}} \dots 1(0)^{\lambda_{2k}-1}$ and is a subsequence of the following string

$$(1, 2, \dots, n)_k.$$

Indeed, each part $1(0)^{\lambda_i}$ gives λ_i increasing blocks $(a_\bullet < a_{\bullet+1}), (a_{\bullet+2}), \dots, (a_{\bullet+\lambda_i})$, so that the last term $a_{\bullet+\lambda_i+1}$ is included in the first increasing block $(a_{\bullet+\lambda_i+1} < a_{\bullet+\lambda_i+2})$ for the part $1(0)^{\lambda_{i+1}}$. There is an exception for $1(0)^{\lambda_{2k}-1}$: the last term a_{4k} has to be counted as a separate increasing block. Hence, the string $a_{2k+2}, a_{2k+3}, \dots, a_{4k}$ consists of

$$\lambda_{k+1} + \lambda_{k+2} + \dots + (\lambda_{2k} - 1) + 1$$

increasing blocks, which is equal to k due to (3).

(An example for $k = 2$ and $n = 8$:

$$\begin{aligned} a_1, a_2, \dots, a_8 &= 1, 5, 7, 6, 3, 4, 8, 2 \\ c_1, c_2, \dots, c_8 &= 1, 1, 0, 0, 1, 1, 0, 0 \\ \lambda_1, \lambda_2, \lambda_3, \lambda_4 &= 0, 2, 0, 2, \end{aligned}$$

where we get two decreasing blocks (5), (7, 6, 3) and two increasing blocks (4, 8), (2).)

Therefore, the string a_1, \dots, a_n is a subsequence of the string

$$(1, n, n-1, \dots, 2)_k (1, 2, \dots, n)_k.$$

This brings a contradiction to the initial assumption and, thus, proves the point **a)** of the theorem.

b) For $n = 4k + 2$, we proceed on the similarly. Assume that a cycle a_1, \dots, a_n is not a subsequence of the cycle

$$(1, 2, \dots, n)_{k+1} (1, n, n-1, \dots, 2)_k.$$

By Lemma 1 with $K = k$ and $M = k + 1$,

$$\lambda_{i+1} + \lambda_{i+2} + \dots + \lambda_{i+k} \leq y - k - 1, \quad \text{for any } 1 \leq i \leq x.$$

Summing all x inequalities, one gets

$$ky = k(\lambda_1 + \lambda_2 + \dots + \lambda_x) \leq x(y - k - 1).$$

The substitutions $y = n - x$ and $n = 4k + 2$ give

$$\begin{aligned} k(n - x) &\leq x(-x + n - k - 1), \\ x^2 - (n - 1)x + kn &\leq 0, \\ x^2 - (4k + 1)x + k(4k + 2) &\leq 0, \\ (x - 2k)(x - (2k + 1)) &\leq 0. \end{aligned}$$

Hence, the integer x must lie in the interval $[2k, 2k + 1]$, that is, either $x = 2k$ or $x = 2k + 1$.

If $x = 2k$ and $y = 2k + 2$, then

$$\lambda_{i+1} + \lambda_{i+2} + \dots + \lambda_{i+k} = y - k - 1 = k + 1, \quad \text{for any } 1 \leq i \leq x.$$

By analogy with **a**), the string a_1, \dots, a_{2k+2} with $a_1 = 1$ has the code $1(0)^{\lambda_1} \dots 1(0)^{\lambda_k}$ and is a subsequence of the string

$$(1, n, n-1, \dots, 2)_k.$$

The code of the string $a_{2k+3}, a_{2k+4}, \dots, a_{4k+2}$ equals $(0)^{\lambda_{k+1}} 1(0)^{\lambda_{k+2}} \dots 1(0)^{\lambda_{2k} - 1}$, where $\lambda_{k+1} + \lambda_{k+2} + \dots + \lambda_{2k} = k + 1$, and so the string occurs as a subsequence in the string

$$(1, 2, \dots, n)_{k+1}.$$

If $x = 2k + 1$ and $y = 2k + 1$, then

$$\lambda_{i+1} + \lambda_{i+2} + \dots + \lambda_{i+k} = y - k - 1 = k, \quad \text{for any } 1 \leq i \leq x,$$

and so $\lambda_{i+k} - \lambda_i = (\lambda_{i+1} + \dots + \lambda_{i+k}) - (\lambda_i + \dots + \lambda_{i+k-1}) = k - k = 0$. Since the integers k and $(2k + 1)$ are coprime, all the numbers $\lambda_1, \lambda_2, \dots, \lambda_{2k+1}$ are equal (to 1). This means that the code of the cycle a_1, \dots, a_n is $(1, 0)_{2k+1} = 1, 0, 1, 0, \dots, 1, 0$. Taking without loss of generality $a_1 = 1$, we have k decreasing blocks $(a_2 > a_3), (a_4 > a_5), \dots, (a_{2k} > a_{2k+1})$ and $(k + 1)$ increasing blocks $(a_{2k+2}), (a_{2k+3} < a_{2k+4}), \dots, (a_{4k+1} < a_{4k+2})$. Therefore, the string $1, a_2, \dots, a_n$ is a subsequence of $(1, n, n-1, \dots, 2)_k (1, 2, \dots, n)_k$.

(An example for $k = 2$ and $n = 10$:

$$\begin{aligned} a_1, a_2, \dots, a_{10} &= 1, 4, 2, 10, 6, 7, 3, 9, 5, 8 \\ c_1, c_2, \dots, c_{10} &= 1, 0, 1, 0, 1, 0, 1, 0, 1, 0 \\ \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 &= 1, 1, 1, 1, 1, \end{aligned}$$

where we get the blocks $(4 > 2), (10 > 6)$ and $(7), (3 < 9), (5 < 8)$, so that the string $\{a_i\}$ occurs in $(1, 10, 9, \dots, 2)_2 (1, 2, \dots, 10)_3$.)

In both cases we have a contradiction to the assumption that a_1, \dots, a_n is not a subsequence of (2) . \square

3. EXAMPLES WHERE $r(n) < \frac{n^2}{2}$ FOR EVEN n

We are going to show that the cycle given in Figure 2 is a rosary of degree 6 and length 17. Indeed, let $a_1, a_2, a_3, a_4, a_5, a_6$ be a permutation of the first 6 positive integers. We will find the permutation among the numbers on the circle, moving in the clockwise direction. It is sufficient to check the 11 cases that are listed in the Table 1.

Here are some other rosaries of degree 6 and length 17:

1, 2, 3, 4, 5, 6, 1, 2, 1, 5, 6, 4, 3, 1, 2, 6, 5;
 1, 2, 3, 4, 5, 6, 1, 2, 3, 5, 6, 4, 3, 2, 1, 6, 5;
 1, 2, 3, 4, 5, 6, 1, 2, 4, 5, 6, 3, 1, 2, 6, 5, 4;
 1, 2, 3, 4, 5, 6, 1, 3, 2, 5, 4, 3, 2, 1, 6, 4, 5;
 1, 2, 3, 4, 5, 6, 1, 3, 4, 2, 1, 6, 5, 4, 3, 5, 6;
 1, 2, 3, 4, 5, 6, 1, 3, 5, 6, 4, 2, 1, 6, 5, 4, 3;
 1, 2, 3, 4, 5, 6, 1, 3, 6, 5, 4, 2, 1, 6, 5, 4, 3;
 1, 2, 3, 4, 5, 6, 1, 4, 3, 2, 1, 6, 5, 2, 3, 4, 6;
 1, 2, 3, 4, 5, 6, 1, 4, 5, 3, 2, 1, 6, 2, 3, 5, 4;
 1, 2, 3, 4, 5, 6, 1, 5, 4, 3, 6, 2, 1, 6, 3, 4, 5.

A long list of such rosaries, containing $(1, 2, 3, 4, 5, 6)$ as a block, was found by a program written by the authors in Java.

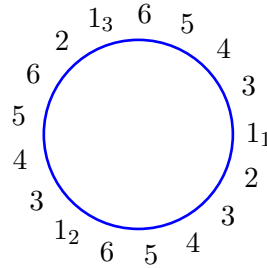


FIGURE 2. A rosary of degree 6.

An example for $n = 8$: the cycle

1, 2, 3, 4, 5, 6, 7, 8, 1, 2, 3, 4, 5, 6, 7, 8, 6, 5, 4, 3, 2, 1, 7, 6, 5, 4, 3, 2, 1, 8, 7

is a rosary of length 31.

TABLE 1. Any permutation of $1, 2, \dots, 6$ is a subsequence of the cycle in Figure 2.

The permutation	Cases	The number on the circle to start with
$1, 2, a_3, a_4, a_5, a_6$	$a_3 < a_4$ or $a_5 > a_6$	1_1
	$a_3 > a_4$ and $a_5 < a_6$	1_2
$1, a_2, 2, a_4, a_5, a_6$	$a_4 > a_5$	1_2
	$a_4 < a_5$	1_3
$1, a_2, a_3, 2, a_5, a_6$	$a_2 < a_3$	1_2
	$a_2 > a_3$	1_3
$1, a_2, a_3, a_4, 2, a_6$	$a_2 < a_3$ or $a_3 < a_4$	1_1
	$a_2 > a_3 > a_4$	1_3
$1, a_2, a_3, a_4, a_5, 2$	$a_2 < a_3$ and $a_4 < a_5$	1_1
	$a_2 < a_3$ and $a_4 > a_5$	1_2
	$a_2 > a_3$	1_3

Question 1. *Do there exist infinitely many even positive integers n for which $r(n) < \frac{n^2}{2}$?*

4. AN UPPER BOUND IN THE CASE THAT n IS ODD

Consider a permutation a_1, \dots, a_n of the first n natural numbers regarded as a cyclic sequence. Let the cycle

$$c_1, \dots, c_n = 1(0)^{\lambda_1} \dots 1(0)^{\lambda_x}$$

be its code consisting of x ones and y zeros, that is,

$$\lambda_1 + \lambda_2 + \dots + \lambda_x = y \quad \text{and} \quad x + y = n.$$

A part $1(0)^{\lambda_i}$ of the cycle c_1, \dots, c_n encodes inequalities

$$a_{\bullet} < a_{\bullet+1} > a_{\bullet+2} > \dots > a_{\bullet+\lambda_i} > a_{\bullet+\lambda_i+1}.$$

An index $i \in [1, x]$ will be called (K, M, N) -lucky if

$$\lambda_{i+1} + \lambda_{i+2} + \dots + \lambda_{i+K} \geq y - M$$

and the number $a_{\bullet+\lambda_i+1}$ does not exceed N .

Lemma 2. *If there exists a (K, M, N) -lucky index i , then the permutation a_1, \dots, a_n is a subsequence of the cycle*

$$(1, 2, \dots, n)_M (1, 2, \dots, N) (n, \dots, 2, 1)_K.$$

Proof of Lemma 2. We have $\lambda_{i+1} + \lambda_{i+2} + \dots + \lambda_{i+K} \geq y - M$ and $a_{\bullet+\lambda_i+1} \leq N$. Then the cycle a_1, \dots, a_n is a juxtaposition of

- ▷ K decreasing blocks $(a_{\bullet+1} > a_{\bullet+2} > \dots > a_{\bullet+\lambda_j} > a_{\bullet+\lambda_j+1})$, where $j \in [i+1, i+K]$, which correspond to

$$(0)^{\lambda_{i+1}} 1(0)^{\lambda_{i+2}} \dots 1(0)^{\lambda_{i+K}},$$

- ▷ at most M increasing blocks $(a_{\bullet} < a_{\bullet+1}), (a_{\bullet+2}), \dots, (a_{\bullet+\lambda_j})$, where $j \in [i+K+1, x] \cup [1, i]$, which correspond to

$$1(0)^{\lambda_{i+K+1}} \dots 1(0)^{\lambda_x} 1(0)^{\lambda_1} \dots 1(0)^{\lambda_i - 1},$$

for the reason that

$$\sum_{j=i+K+1}^x \lambda_j + \sum_{j=1}^i \lambda_j = y - \sum_{j=i+1}^{i+K} \lambda_j \leq M,$$

- ▷ one increasing block $(a_{\bullet+\lambda_i+1})$, where $a_{\bullet+\lambda_i+1} \leq N$.

Hence, the cycle $(1, 2, \dots, n)_M (1, 2, \dots, N) (n, \dots, 2, 1)_K$ contains the permutation a_1, \dots, a_n as a subsequence. \square

The following theorem provides rosaries of odd degrees, its proof is a refinement of the method for Theorem 2.

Theorem 3. *Let k be a positive integer.*

- a) *If $n = 4k + 1$, then the sequence*

$$(1, 2, \dots, n)_k (1, 2, \dots, 3k) (n, \dots, 2, 1)_{k-1} (n, \dots, 2) \quad (4)$$

is a rosary of degree n .

- b) *If $n = 4k + 3$, then the sequence*

$$(1, 2, \dots, n)_k (1, 2, \dots, 3k+2) (n, \dots, 2, 1)_k (n, \dots, 2) \quad (5)$$

is a rosary of degree n .

Proof. a) Let us assume that the cycle a_1, \dots, a_n is not a subsequence of (4). Applying Lemma 1 with $K = M = k$, we obtain

$$\lambda_{i+1} + \lambda_{i+2} + \dots + \lambda_{i+k} \leq y - k, \quad \text{for each } 1 \leq i \leq x.$$

Denote by $E \subset \{1, \dots, x\}$ the set of indices i such that the following equality holds

$$\lambda_{i+1} + \lambda_{i+2} + \dots + \lambda_{i+k} = y - k,$$

and so for all $i \notin E$, we have the inequality

$$\lambda_{i+1} + \lambda_{i+2} + \dots + \lambda_{i+k} \leq y - k - 1.$$

Let q be the cardinal of the set E . Summing all q equalities and $(x - q)$ inequalities gives

$$\begin{aligned} ky &= (\lambda_1 + \lambda_2 + \dots + \lambda_k) + (\lambda_2 + \lambda_3 + \dots + \lambda_{k+1}) + \dots + (\lambda_x + \lambda_1 + \dots + \lambda_{k-1}) \\ &\leq q \cdot (y - k) + (x - q) \cdot (y - k - 1) = x(y - k) - (x - q). \end{aligned}$$

Since $y = n - x$, we get

$$\begin{aligned} k(n - x) &\leq x(-x + n - k) - (x - q), \\ x^2 - (n - 1)x + kn - q &\leq 0. \end{aligned}$$

Therefore, the discriminant of the polynomial $t^2 - (n - 1)t + kn - q$ must be nonnegative:

$$\mathcal{D} = (n - 1)^2 - 4(kn - q) = (4k)^2 - 4k(4k + 1) + 4q = 4q - 4k \geq 0,$$

that is, $q \geq k$.

Consider the set of the last terms $a_{\bullet+\lambda_i+1}$ of the maximal decreasing blocks

$$a_{\bullet+1} > a_{\bullet+2} > \dots > a_{\bullet+\lambda_i} > a_{\bullet+\lambda_i+1}$$

encoded by the parts $(0)^{\lambda_i}$ of c_1, \dots, c_n where $i \in E$. The numbers $a_{\bullet+\lambda_i+1}$ are different and don't exceed $n = 4k + 1$. Moreover, such a number cannot be equal to n , else the block $a_{\bullet+1} > \dots > a_{\bullet+\lambda_i+1}$ wouldn't be maximal.

We shall now investigate the cases where $q > k$ and $q = k$ separately.

Case $q > k$. Since the interval $[3k+1, 4k]$ contains k integers and $|E| = q \geq k + 1$, there exists an index $e \in E$ such that $a_{\bullet+\lambda_e+1} \leq 3k$. Then the index e is $(k, k, 3k)$ -lucky, and so according to Lemma 2, the permutation a_1, \dots, a_n is a subsequence of the cycle (4).

Case $q = k$. The inequality $x^2 - (n - 1)x + kn - q = (x - 2k)^2 \leq 0$ implies that $x = 2k$, and so $y = 2k + 1$. For exactly $k = |E|$ values of $i \in \{1, 2, \dots, 2k\}$, we have the equality

$$\lambda_{i+1} + \lambda_{i+2} + \dots + \lambda_{i+k} = k + 1,$$

For others, the equality $\lambda_{i+1} + \lambda_{i+2} + \dots + \lambda_{i+k} = k$ takes place, since $\lambda_1 + \lambda_2 + \dots + \lambda_{2k} = 2k + 1$. Suppose that $a_{\bullet+\lambda_i+1} > 3k$ for all indices $i \in E$. Counting the number of integers in the maximal decreasing blocks $a_{\bullet+1} > \dots > a_{\bullet+\lambda_i+1}$, we conclude that there are at least

$$\sum_{i \in E} (\lambda_i + 1), \quad \text{i.e. } k + \sum_{i \in E} \lambda_i, \quad (6)$$

integers in the interval $[3k+1, 4k+1]$. Hence, we have $\sum_{i \in E} \lambda_i \leq 1$. Let $u \in E$ be an index such that $u+1 \notin E$ (modulo $2k$). The number $a_{\bullet+\lambda_u+2}$ is not counted in (6), and yet $3k < a_{\bullet+\lambda_u+1} < a_{\bullet+\lambda_u+2}$. If $\sum_{i \in E} \lambda_i = 1$, then the interval $[3k+1, 4k+1]$ contains at least

$$\left(k + \sum_{i \in E} \lambda_i \right) + 1, \quad \text{i.e. } k + 2,$$

integers, which is impossible.

If $\sum_{i \in E} \lambda_i = 0$, then there are two distinct indices $u, v \in E$ such that $u+1, v+1 \notin E$ (modulo $2k$), otherwise, the set E would be a subset of k consecutive elements of the cycle

$1, 2, \dots, 2k$, and so $\sum_{i \in E} \lambda_i \geq k > 0$. We thus get two numbers $a_{\bullet+\lambda_u+2}$ and $a_{\bullet+\lambda_v+2}$ for which $3k < a_{\bullet+\lambda_u+1} < a_{\bullet+\lambda_u+2}$ and $3k < a_{\bullet+\lambda_v+1} < a_{\bullet+\lambda_v+2}$. Together with (6), this gives

$$\left(k + \sum_{i \in E} \lambda_i \right) + 2, \text{ i.e. } k + 2,$$

distinct integers in the interval $[3k+1, 4k+1]$, which is impossible. Therefore, $a_{\bullet+\lambda_i+1} \leq 3k$ for at least one index $i \in E$. Such an index is $(k, k, 3k)$ -lucky, and according to Lemma 2, the permutation a_1, \dots, a_n is a subsequence of the cycle (4).

In both cases we obtain a contradiction to the initial assumption that a_1, \dots, a_n is not a subsequence of the cycle (4).

b) Assume that the permutation a_1, \dots, a_n is not a subsequence of the cycle (5). According to Lemma 1 with $K = k + 1$ and $M = k$, we have

$$\lambda_{i+1} + \lambda_{i+2} + \dots + \lambda_{i+k+1} \leq y - k, \quad \text{for each } 1 \leq i \leq x.$$

Let $E \subset \{1, \dots, x\}$ be the set of indices i such that

$$\lambda_{i+1} + \lambda_{i+2} + \dots + \lambda_{i+k+1} = y - k.$$

Thus, for any $i \notin E$, the following inequality holds

$$\lambda_{i+1} + \lambda_{i+2} + \dots + \lambda_{i+k+1} \leq y - k - 1.$$

Summing all $q (= |E|)$ equalities and $(x - q)$ inequalities gives

$$\begin{aligned} (k+1)y &= (\lambda_1 + \lambda_2 + \dots + \lambda_{k+1}) + (\lambda_2 + \lambda_3 + \dots + \lambda_{k+2}) + \dots + (\lambda_x + \lambda_1 + \dots + \lambda_k) \\ &\leq q \cdot (y - k) + (x - q) \cdot (y - k - 1) = x(y - k) - (x - q), \end{aligned}$$

Since $y = n - x$, we get

$$\begin{aligned} (k+1)(n - x) &\leq x(-x + n - k) - (x - q), \\ x^2 - nx + (k+1)n - q &\leq 0. \end{aligned}$$

Therefore, the discriminant of the polynomial $t^2 - nt + (k+1)n - q$ must be nonnegative:

$$\mathcal{D} = n^2 - 4(k+1)n + 4q = (4k+3)^2 - 4(k+1)(4k+3) + 4q = 4q - (4k+3) \geq 0,$$

that is, $q \geq k+1$ as $q \in \mathbb{Z}$. We have q indices $i \in E$ for which

$$\lambda_{i+1} + \lambda_{i+2} + \dots + \lambda_{i+k+1} = y - k.$$

Consider the set of the last terms $a_{\bullet+\lambda_i+1}$ of the maximal decreasing blocks

$$a_{\bullet+1} > a_{\bullet+2} > \dots > a_{\bullet+\lambda_i} > a_{\bullet+\lambda_i+1}$$

encoded by the parts $(0)^{\lambda_i}$ of c_1, \dots, c_n where $i \in E$. The numbers $a_{\bullet+\lambda_i+1}$ are different and don't exceed $n = 4k + 3$. Moreover, such a number cannot be equal to n , else the block $a_{\bullet+1} > a_{\bullet+2} > \dots > a_{\bullet+\lambda_i} > a_{\bullet+\lambda_i+1}$ wouldn't be maximal. Therefore, since $q \geq k+1$, there exists an index $e \in E$ such that $a_{\bullet+\lambda_e+1} \leq 3k+2$. Then the index e is $(k+1, k, 3k+2)$ -lucky, and so by Lemma 2, the permutation a_1, \dots, a_n is a subsequence of the cycle (5).

This contradiction to the initial assumption proves the statement of the theorem. \square

For $n = 4k + 1$, the length of the rosary (4) of degree n is equal to

$$2k \cdot n + 3k - 1 = 8k^2 + 5k - 1 < \frac{16k^2 + 8k + 1}{2} + \frac{4k + 1}{4} - 1 = \frac{n^2}{2} + \frac{n}{4} - 1.$$

For $n = 4k + 3$, the length of the rosary (5) of degree n is equal to

$$(2k + 1) \cdot n + 3k + 1 = 8k^2 + 13k + 4 < \frac{16k^2 + 24k + 9}{2} + \frac{4k + 3}{4} - 1 = \frac{n^2}{2} + \frac{n}{4} - 1.$$

Corollary 1. *If $n > 1$ is odd, then $r(n) < \frac{n^2}{2} + \frac{n}{4} - 1$.*

5. EXAMPLES FOR ODD n

Let us now bring examples showing that one shouldn't expect obtaining the inequality $r(n) \leq \frac{n^2}{2}$ by improving the construction of Theorem 3.

Consider the permutation given in Figure 3. It has 10 maximal increasing blocks and 11

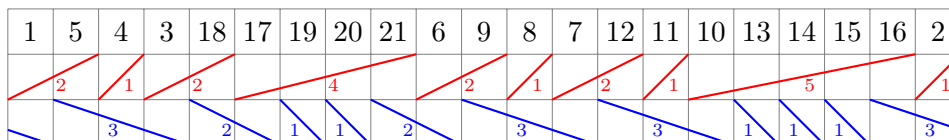


FIGURE 3. A permutation of the numbers $1, 2, \dots, 21$.

maximal decreasing blocks (they are illustrated by the ascending and the descending segments respectively). By trying to arrange 5 consecutive increasing blocks of the cycle into the string $(1, 2, \dots, 21)_5$ so that remaining part would fit in the string $(1, 2, \dots, 12)(21, \dots, 2, 1)_5$, one receives evidence that the permutation is not a subsequence of the cycle

$$(1, 2, \dots, 21)_5 (1, 2, \dots, 12)(21, \dots, 2, 1)_4 (21, \dots, 2)$$

of length $221 = \lceil 21^2/2 \rceil + 1$. Thus, in the statement of Theorem 3 with $k = 5$ and $n = 21$, we couldn't have replaced the block $(1, 2, \dots, 3k)$ by the block $(1, 2, \dots, 2k+1)$ in order to obtain the estimation $r(n) \leq \frac{n^2}{2}$.

An example for $k = 8$ and $n = 33$: the permutation

$$1, 5, 4, 3, 2, 7, 6, 11, 10, 9, 8, 18, 19, 20, 21, 22, 23, 14, 13, 12, 25, \\ 24, 26, 17, 16, 15, 28, 27, 29, 30, 31, 32, 33$$

is not a subsequence of the cycle $(1, 2, \dots, 33)_8 (1, 2, \dots, 17)(33, \dots, 2, 1)_7 (33, \dots, 2)$ of length $544 = \lceil 33^2/2 \rceil$.

Question 2. *What is the value of $r(n)$ when $n = 21$ or 33 ?*

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